Euler's Method in Euler's Words

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April 4, 2007

Introduction

Euler's method is a technique for finding approximate solutions to differential equations addressed in a number of undergraduate mathematics courses. Representative texts addressing Euler's method for calculus [4], differential equations [1], mathematical modeling [8], and numerical methods [2] courses are identified in the references. Each of those courses are opportunities to give students an opportunity to read Euler's description of the algorithm, and in the process come to understand the technique and its shortcomings from Euler's own words. This chapter includes historical information about Euler and his mathematics at the time when he wrote about the approximation method. Additionally, student activities are offered that will connect that history to the mathematics the students are learning.

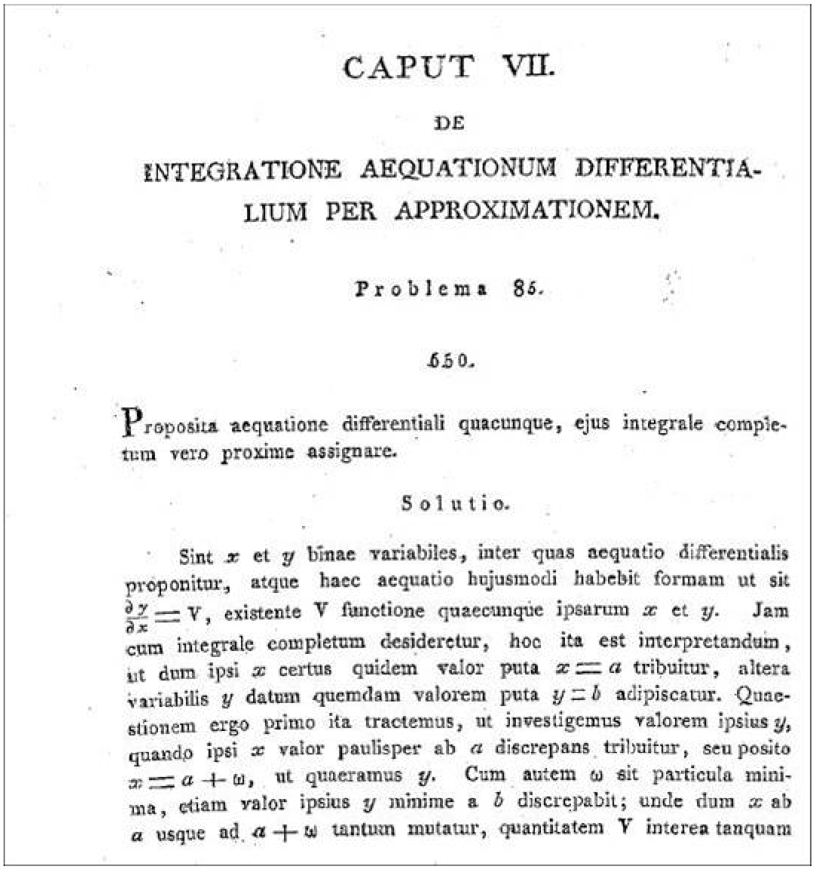
Historical preliminaries

Leonhard Euler (1707-1783) was one of the most gifted of all mathematicians. Excellent biographies of Euler, some identifying the voluminous quantity of his mathematical writing, are identified in the Annotated Bibliography found in Appendix A. One of Euler's many gifts was his ability to write mathematics clearly and understandably. The great French mathematician Pierre-Simon Laplace (1749-1827) commented on Euler's writing, "Read Euler, read Euler. He is the master of us all."[5] It is not unreasonable to believe that our students can find Euler readable, particularly Euler's textbooks on the calculus.

Leonhard Euler (1707-1783)

While in the service of the Russian Empress Catherine the Great, Euler published a text on the integral calculus, *Institutionum calculi integralis* [7], part of which appears on the next page. Euler wrote at least some of this volume when he was at the Berlin Academy and employed by Frederick the Great of Prussia, prior to Euler's return to the St. Petersburg Academy in 1766. Previously, Euler had written precursors to this three volume integral calculus text. In 1755, he published *Institutiones Calculi Differentialis*, his text on the differential calculus. His "precalculus" book, *Introductio in Analysin Infinitorum*, was published in 1748.

Since the calculus had only been discovered within a hundred years of these publications by Euler, his texts were among the first on this relatively new mathematics. Euler read the works of inventors of the calculus, Sir Isaac Newton (1643-1727) and Gottfried Leibniz (1646-1714), as well as those of their respective disciples, to include Brook Taylor (1685-1731) and Johann Bernoulli (1667-1748). Euler adopted the best of their notation, overlooked the worst, and included many of his own innovations. Euler's texts were widely read by his peers and successors, and the notation and terminology we use today in our undergraduate calculus and differential equations texts are largely due to Euler.


CAPUT VII.

DE

INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM PER APPROXIMATIONEM.

Problema 85.

650.

Proposita aequatione differentiali quacunque, ejus integrale completum vero proxime assignare.

Solutio.

Sint x et y binae variabiles, inter quas aequatio differentialis proponitur, atque haec aequatio hujusmodi habebit formam ut sit $\frac{\partial y}{\partial x} = V$, existente V functione quaecunque ipsarum x et y. Jam cum integrale completum desideretur, hoc ita est interpretandum, ut dum ipsi x certus quidem valor puta $x = a$ tribuitur, altera variabilis y datum quemdam valorem puta $y = b$ adipiscatur. Quaestionem ergo primo ita tractemus, ut investigemus valorem ipsius y, quando ipsi x valor paulisper ab a discrepans tribuitur, seu posito $x = a + \omega$, ut quaeramus y. Cum autem ω sit particula minima, etiam valor ipsius y minime a b discrepabit; unde dum x ab a usque ad $a + \omega$ tantum mutatur, quantitatem V interea tanquam

The start of Chapter 7 of *Institutionum calculi integralis*, courtesy of The Euler Archive.

Euler writes about differential equations in his *Foundations of Differential Calculus*, in which he states that "...the main concern of integral calculus is the solution of differential equations..."[6]. Largely because of his coverage of solutions to differential equations, it took three volumes for Euler to address the integral calculus, while the differential calculus was covered in just one.

Our attention is on the first of those three volumes on the integral calculus. The first section of that first volume is on integral formulas; the remaining two sections of the book are on the solution of differential equations. Depicted above is the beginning of the seventh chapter of the second section, which is translated in Appendix C to this chapter. After some mathematical preliminaries, we describe these opening paragraphs of chapter 7, as they contain the algorithm that we call Euler's method. Euler obtained exact solutions for many differential equations in his calculus text, but he acknowledged that there were many differential equations for which the best he could do was obtain an approximation to the exact solution. Chapter 7 begins with Euler's description of his algorithm to approximate solutions, and continues with improvements, to include the use of power series to solve differential equations.

Mathematical preliminaries: Euler's method

Euler's method is a crude method for approximating solutions to differential equations. It is crude for reasons that Euler explains in the corollaries contained in the first couple of pages of chapter 7 of *Institutionum calculi integralis*. We discuss those later. In this section we briefly review Euler's method and provide an example of its application. More details can be found in a variety of texts [1, 8, 2].

We start with a first order differential equation, $\frac{dy}{dt} = f(t, y)$, with initial values $y(t_0) = y_0$. The differential equation is converted to a difference equation, $y_{k+1} = y_k + \Delta t f(t_k, y_k)$, where $t_{k+1} = t_k + \Delta t$ with step size Δt. Approximate solutions are computed recursively starting with the known initial value $y_0 =$

$y(t_0)$. Euler provides one method for deriving the recursive relationship; we provide other methods in Appendix B.

Approximation methods should be used when a method that provides an exact solution is not available. It is often best to demonstrate Euler's method on a differential equation that students cannot integrate using elementary functions, such as the initial value problem,

$$\frac{dy}{dt} = e^{-t^2}$$

with $y(0) = 1$. The corresponding Euler's method difference equation is

$$y_{k+1} = y_k + \Delta t e^{-t_k^2}$$

and starting value $y_0 = 1$. Successive iterations of the difference equation are readily calculated, using a step size $\Delta t = 0.10$:

$$y_1 = y_0 + \Delta t e^{-t_0^2} = 1 + (0.10)(e^{-0^2}) = 1.10$$

$$y_2 = y_1 + \Delta t e^{-t_1^2} = 1.10 + (0.10)(e^{-0.10^2}) = 1.20$$

$$y_3 = y_2 + \Delta t e^{-t_2^2} = 1.20 + (0.10)(e^{-0.20^2}) = 1.30$$

$$\dots$$

These iterates are readily tabulated:

t_k	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
y_k	1.00	1.10	1.20	1.30	1.39	1.47	1.55	1.62	1.68	1.73	1.78

Euler describes just such a display, as we will soon discuss, though he does not provide a specific numerical example in chapter 7.

Euler's description of the method

In sections 650 to 653 of chapter 7 (see Appendix C for a translation), Euler described the algorithm for obtaining an approximation to the solution of a first order differential equation. In section 650 he derived and discussed the implementation; in the remaining three sections he provided a summary and offered warnings about the error associated with the method.

The differential equation that Euler solved has the form $\frac{dy}{dx} = V(x, y)$, with initial values $x = a$ and $y = b$. His goal was to incrementally find the value of y when x changed just a little, or when $x = a + \omega$ in his notation. In our notation, we write $x_{k+1} = x_k + \Delta x$, with $x_0 = a$, and $\omega = \Delta x$. Euler made the assumption that $V(x, y)$ was constant in the small interval, or $A = V(a, b)$. He then integrated the resulting differential equation $\frac{dy}{dx} = A$ and found the value of the integration constant so that the solution satisfied the initial data $x = a$ and $y = b$. When he evaluated the resulting solution $y = b + A(x - a)$ at $x = a + \omega$, Euler obtained

$$y = b + A\omega.$$

That last equation is equivalent to the recursion formula, in our modern notation, $y_{k+1} = y_k + \Delta x f(x_k, y_k)$. Euler obtained the next x using $x = a + \omega$, or $a' = a + \omega$. He found the next y using $y = b + A\omega$, or $b' = b + A\omega$. The value b' was Euler's approximate solution to the differential equation at $x = a'$. Euler computed the next approximation by first evaluating $V(a', b')$ to obtain A', and then substituted the new values into $y = b + A\omega$ to get $b'' = b' + A'\omega$, where b'' is the numerical solution at $x = a' + \omega = a''$. Euler then repeated this process iteratively, obtaining approximate values for the solution as far from the initial values as he desired.

Section 651 is the first corollary, in which Euler reiterated that successive values of x and y are obtained by repeated calculutions. In corollary two, Euler pointed out that the error can be reduced by making the incremental steps small, but even with that, the error accumulates. In the third corollary Euler stated that not only does the error depend on the step-size, but also on the variability of the function $V(x, y)$ in the interval. He specified that if $V(x, y)$ varies greatly in the interval, the error of the approximation is large. In those corollaries, Euler articulated key ideas concerning numerical methods, which an instructor today can use to focus student learning on these important concepts.

Student activities

Instructors can engage students in a variety of learning activities using Euler's description of the algorithm used to approximate solutions to differential equations. Some of those activities are described here, which are offered for consideration and modification at the instructor's discretion.

Since the translation (see Appendix C) is brief, assigning the translation as a student reading assignment is a place to start. This assignment can supplement students' reading of the corresponding section of the course text, and the following questions can be provided to guide student reading the translation. The questions can also be used for student homework, in-class activities, or writing assignments. The questions help students learn to become "active readers" of Euler's text. As we all know, reading a mathematics text is different than reading a text in other disciplines.

- What words would we use to describe what Euler meant by the expression "a complete integral"?
- Euler uses the notation $\frac{\partial y}{\partial x} = V$ for the differential equation. What would we use? Why might the partial derivative notation be appropriate?
- How does Euler justify transforming the differential equation $\frac{\partial y}{\partial x} = V$ to $\frac{\partial y}{\partial x} = A$?
- If we know $V(x, y)$, how do we determine the value of the constant that Euler labels A?

- How does Euler arrive at $y = b + A(x-a)$? Show the steps necessary to arrive at the solution $y = b + A(x-a)$ to the differential equation $\frac{\partial y}{\partial x} = A$ and the given initial data.

- Euler uses the Greek letter ω to represent a small quantity. What is the corresponding parameter in the version of Euler's method described in our course text?

- Which equation in the translation is Euler's version of the difference equation $y_{k+1} = y_k + \Delta t f(t_k, y_k)$?

- What is the point Euler is trying to make in Corollary 1?

- In Corollary 2, what are the two significant points about the error made in implementing this algorithm?

- What does Corollary 3 state about the relationship of the function V and the error of the algorithm?

One way for students to process the translation is to have them read the translation in class as an entire-class activity. Going around the room, each student reads one sentence of Euler, and explains what she just read. Other members of the class can comment as they'd like on the interpretation. Then the next student reads the next sentence of Euler, and he explains what he just read, with others adding to the discussion as appropriate. The process is repeated until the reading is completed. Historians of mathematics read original sources in this manner, as exemplified by the Arithmos (www.arithmos.org) reading group in the northeast and Oresme (www.nku.edu/~curtin/oresme.html) in the midwest.

In completing the reading and answering these questions, students will obtain a deeper understanding of Euler's method than they would by simply reading the course text or passively listening to a lecture on the topic. There are more student activities in Appendix B, and some of those activities may be used as a classroom activity or as out-of-class student projects, with students working individually or in groups as the instructor prefers.

Summary and conclusion

Reading Euler's introduction to methods for approximating the solution of differential equations can be a meaningful activity for students learning Euler's method. By learning from the "master of us all," students will gain an understanding of the origins of the method and an understanding of why this mathematical method was invented. Additionally, they will gain an appreciation for our modern notation and its origins. Most importantly, Euler clearly describes some of the important practices and cautions to be observed in implementing the method, which should deepen student understanding of the algorithm if they actively read Euler's work.

References

[1] Paul Blanchard, Robert L. Devaney, Glen R. Hall, *Differential Equations,* 3rd edition, Brooks/Cole, Pacific Grove, CA, 2006.

[2] Richard Burden and J. D. Faires, *Numerical Analysis*, 7th ed., Brooks/Cole, Pacific Grove, CA, 2001.

[3] Jean-Luc Chabert, et al., *A History of Algorithms: From the Pebble to the Microchip,* Springer, Berlin, 1999.

[4] David W. Cohen and James M. Henle, *Calculus: The Language of Change,* Jones and Bartlett, Sudbury, MA, 2005.

[5] William Dunham, *Euler: The Master of Us All*, The Mathematical Association of America, Washington, 1999, p. xiii.

[6] Leonhard Euler, *Foundations of Differential Calculus*, translated by John D. Blanton, Springer, New York, 2000, p. 167.

[7] Leonhard Euler, *Institutionum Calculi integralis,* vol. I, St. Petersburg, 1768. Available from The Euler Archive (www.eulerarchive.org).

[8] Frank R. Giordano, Maurice D. Weir, William P. Fox, *A First Course in Mathematical Modeling*, Brooks/Cole, Pacific Grove, CA, 2003.

[9] Herman Goldstine, *A History of Numerical Analysis From the 16th Through the 19th Century,* Springer, New York, 1977.

Appendix A: Annotated bibliography

Jean-Luc Chabert, et al., *A History of Algorithms: From the Pebble to the Microchip,* Springer, Berlin, 1999. This is a comprehensive resource for those interested in the history of calculating from Babylonian methods, through Chinese counting tables, Napier's rods, Gaussian elimination and more. Portions of the original sources are provided and translated in English; a translation of the relevant portion of Euler's work describing Euler's method is an example of just such a translation.

William Dunham, *Euler: The Master of Us All*, The Mathematical Association of America, Washington, 1999. This superb book contains a brief biographical sketch and eight chapters explaining the work of Euler. The chapters are on subjects found in undergraduate mathematics curricula. Personally, I have made good use of the contents of the chapter on complex variables in my complex variables course. There is no discussion of Euler's method in this book, however.

The Euler Archive (www.eulerarchive.org) contains original works by Euler and translations. Additionally, this web site contains biographical information on Euler and historical information about the times in which Euler lived.

Leonhard Euler, *Institutionum Calculi integralis,* vol. I, St. Petersburg, 1768, available in The Euler Archive . This is the original work, found in The Euler Archive by searching on Enestrom number 342. Euler's method is found in the Second Section, Chapter 7. Although in Latin, students can still gain an

appreciation of the work of the master in viewing his original text. Chapter 7 contains far more than the introductory material that describes Euler's method; the chapter continues and describes more accurate formulas using power series expansions.

Herman Goldstine, *A History of Numerical Analysis From the 16th Through the 19th Century,* Springer, New York, 1977. This out-of-print text is more limited than the Chabert book in that it not only is more restricted in time but also is Euro-centric in its coverage. But the depth and level of detail is far more extensive on the topics that it covers, and it covers a variety of methods that Chabert, et al., chose not to include in their later book.

John J. O'Connor and Edmund F. Robertson, founders, *The MacTutor History of Mathematics Archive*, (http://www-history.mcs.st-andrews.ac.uk/index.html), University of St. Andrews, Scotland. This comprehensive history of mathematics web site contains significant information about Euler and his mathematical work.

Appendix B: Student assignments

Listed here are general descriptions of some additional activities for students, which instructors can consider adapting for their courses. In every case, depending on how the instructor would like to implement these activities, an appropriate level of detail would have to be added. For example, in the first bulleted assignment, the instructor may specify the step-size, the differential equations, and the technology students are to use in completing the assignment.

The goal of the assignments is that in completing these activities, students will have a deeper understanding of Euler's method and the associated mathematics. An additional assignment, more appropriate for a senior seminar or an academic conference, would be for students to continue the translation of subsequent sections of Chapter 7. Such a translation could be submitted for publication at The Euler Archive (www.eulerarchive.org).

- Euler does not provide a specific example of the method in this chapter of his text. Choose an appropriate differential equation, approximate the solution as Euler describes, and create a table similar to that found in the translation but displaying actual numerical values.

- The translator tried to retain the punctuation, capitalization and vocabulary used by Euler. Rewrite the translation using the notation and language that you find in our course text.

- Implement Euler's method with differential equations for which you can determine the exact solution. Use your examples to demonstrate each of the points that Euler makes about error in the last two corollaries.

- Euler derived the relationship used for iteration, $y = b + A\omega$, by solving the general form for the differential equation. Our course text derives the relationship using a graphical method. Explain each of the steps in the following alternative methods for deriving Euler's method:

a.) Using the definition of the derivative and given the differential equation $\frac{dy}{dt} = f(t, y)$:

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{y(t + \Delta t) - y(t)}{\Delta t}$$

$$\frac{dy}{dt} \approx \frac{y(t + \Delta t) - y(t)}{\Delta t}$$

$$\frac{dy}{dt} = f(t, y) \approx \frac{y(t + \Delta t) - y(t)}{\Delta t}$$

$$y(t + \Delta t) \approx y(t) + \Delta t f(t, y)$$

$$y_{k+1} = y_k + \Delta t f(t_k, y_k)$$

b.) Using Taylor series and given the same differential equation:

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{y''(t_0)(t - t_0)^2}{2} + \cdots$$

$$y(t_0 + \Delta t) = y(t_0) + y'(t_0)(\Delta t) + \frac{y''(t_0)(\Delta t)^2}{2} + \cdots$$

$$y(t_0 + \Delta t) = y(t_0) + \frac{dy}{dt}(t_0)(\Delta t) + \frac{y''(t_0)(\Delta t)^2}{2} + \cdots$$

$$y(t_0 + \Delta t) = y(t_0) + \Delta t f(t_0, y_0) + \frac{y''(t_0)(\Delta t)^2}{2} + \cdots$$

$$y_{k+1} = y_k + \Delta t f(t_k, y_k)$$

c.) Explain how the derivation using Taylor series may be used to obtain an estimate for the error involved in implementing Euler's method.

Appendix C: Original source translation

This is the author's translation of the initial sections of Leonhard Euler's, *Institutionum calculi integralis,* vol. I, St. Petersburg, 1768, posted in The Euler Archive. The punctuation and notation of the original were retained in this translation, which sometimes makes the translation seem awkward.

CHAPTER VII

ON

THE INTEGRATION OF DIFFERENTIAL EQUATIONS BY APPROXIMATION

Problem 85.

650.

Whenever presented a differential equation, find its complete integral very approximately.

Solution

The pair of variables x and y appear in a differential equation, and moreover this equation has the form $\frac{\partial y}{\partial x} = V$, the function V itself a function of x and y. We desire the complete integral, which is interpreted that as long as x is assigned a certain value $x = a$, the other variable y takes on a given value $y = b$. Therefore our primary goal is to find the value of y so that when x takes on a value that differs little from a, or we assume $x = a + \omega$, then we can find y. Since ω is a very small quantity, then the value of y itself differs minimally from b; so while x varies a little from a to $a+\omega$, one may consider the quantity V as a constant. When we specify $x = a$ and $y = b$ then $V = A$, and by virtue of the small change we have $\frac{\partial y}{\partial x} = A$, for that reason when integrating $y = b + A(x-a)$, a constant being added of course, so that when $x = a$ we have $y = b$. Therefore given the initial values $x = a$ and $y = b$, we obtain the approximate next values $x = a + \omega$ and $y = b + A\omega$, so that proceeding further in a similar way over the small interval, in the end arriving at values as distant as we would like from the earlier values. These operations can be placed for ease of viewing, displayed successively in the following manner.

Variable	successive values
x	$a,\ a',\ a'',\ a''',\ a^{IV},\ \ldots\ {}'x,\ x$
y	$b,\ b',\ b'',\ b''',\ b^{IV},\ \ldots\ {}'y,\ y$
V	$A,\ A',\ A'',\ A''',\ A^{IV},\ \ldots\ {}'V,\ V$

Certainly from the given initial values $x = a$ and $y = b$, we have $V = A$, then for the second we have $b' = b + A(a' - a)$, the difference $a' - a$ as small as one pleases. From here in putting $x = a'$ and $y = b'$, we obtain $V = A'$, and from this we will obtain the third $b'' = b' + A'(a'' - a')$, when we put $x = a''$ and $y = b''$, we obtain $V = A''$. Now for the fourth, we have $b''' = b'' + A''(a''' - a'')$, from this, placing $x = a'''$ and $y = b'''$, we shall obtain $V = A'''$, thus we can progress to values as distant from the initial values as we wish. The first sequence of x values can be produced successively as desired, provided it is ascending or descending over very small intervals.

Corollary 1.

651. Therefore one at a time over very small intervals calculations are made in the same way, so the values, on which the next depend, are obtained. As values of x are done iteratively in this way one at a time, the corresponding values of y are obtained.

Corollary 2.

652. Where smaller intervals are taken, through which the values of x progress iteratively, so much the more accurate values are obtained one at a time. However the errors committed one at a time, even if they may be very small, accumulate because of the multitude.

Corollary 3.

653. Moreover errors in the calculations arise, because in the individual intervals the quantities x and y are seen to be constant, so we consider the function V as a constant. Therefore the more the value of V changes on the next interval, so much the more we are to fear larger errors.